

# Generating functional for the full parquet approximation

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Parquet diagrams sum self-consistently Feynman graphs for the vertex function with all two-particle multiple scatterings. We show how the complete parquet equations for the Hubbard-like models can be integrated to a generating functional from which all thermodynamic quantities are derived via (functional) derivatives. An explicit Luttinger-Ward functional  $\Phi[G; \Lambda, \mathcal{K}; U]$  is constructed containing only the renormalized one-particle,  $G$ , irreducible,  $\Lambda$ , and reducible,  $\mathcal{K}$ , two-particle propagators as independent variational functions. The parquet approximation is proven to be a thermodynamically consistent,  $\Phi$ -derivable theory obeying the necessary conservation laws.

One of the major problems in the theory correlated electrons is to construct in a systematic and controlled way a consistent approximation interpolating reliably between weak- and strong-coupling regimes. The two extreme limits of weak and strong couplings in the archetypal Hubbard model can be described relatively well. The weak-coupling regime is governed by a Hartree-Fock mean field with dynamical fluctuations covered by Fermi-liquid theory. Extended systems at low temperatures are Pauli paramagnets with smeared out local magnetic moments. For bipartite lattices antiferromagnetic long-range order sets in at half filling and zero temperature at arbitrarily small interaction. In the strong-coupling regime the Hubbard model at half filling maps onto a Heisenberg antiferromagnet with pronounced local magnetic moments and the Curie-Weiss law for the staggered magnetic susceptibility, at least at the mean-field level. The spectral structure is dominated by separated lower and upper Hubbard bands and the strongly correlated system seems to be insulating even in the paramagnetic phase.

However, it is the intermediate coupling, where the effective Coulomb repulsion is comparable with the kinetic energy and hence neither very weak nor very strong, that is of great interest for the theorists as well as for the experimentalists. At intermediate coupling dynamical fluctuations control the low-temperature physics of interacting electrons and neither weak-coupling nor atomic-like perturbation theories are adequate. In this nonperturbative regime a singularity in a generic two-particle function is approached and we expect breakdown of the Fermi-liquid regime and a transition to an ordered state or eventually to a Mott insulator.

Unfortunately there are only a few techniques for a quantitative analysis of the transition region between weak and strong coupling. Exact methods such as the Bethe ansatz or the numerical renormalization group can be applied only to one-dimensional or single-impurity models [1,2]. Numerical quantum Monte Carlo is good for thermodynamic properties at relatively high temperatures and is restricted to small samples [3]. Analytic methods are mostly of effective or mean-field nature [4]. Systematic diagrammatic expansions usually do not go beyond the fluctuation-exchange approximation (FLEX) that is known to fail at intermediate coupling [5]. Although some improvements upon or alternatives to the FLEX approximation have been proposed, the vertex renormalization in these theories remain static [6,7]. The full dynamic and self-consistent vertex renormalization is contained first in the parquet diagrams [8].

Parquet diagrams were introduced to describe interaction of mesons more effectively [9]. Since then a number of attempts have been made to utilize the nontrivial renormalization scheme of the parquet algebra in condensed matter. Kondo effect [10], x-ray edge problem [11], formation of the local magnetic moment [12] are among the most well known applications. Inability to solve the parquet equations effectively has impeded broader application of the method.

Parquet diagrams represent a systematic way of summation and renormalization of Feynman graphs. Instead of concentrating on the one-particle irreducible diagrams and the Dyson equation, the parquet approach sums two-particle diagrams contributing to vertex functions for which Bethe-Salpeter equations are constructed. The resulting algebra is much more complicated than that of the one-particle approximations. The two-particle Green functions obtained from the parquet approximation are suitable for spectral properties of the system, but it is cumbersome to gain thermodynamic properties out of them. It has been hitherto unclear whether the parquet approximation forms a thermodynamically consistent  $\Phi$ -

derivable approximation fulfilling the necessary conservation laws [13]. Only if we construct a generating Luttinger-Ward functional in closed form from which all the thermodynamic quantities can be derived via derivatives with respect to auxiliary sources, we can be sure thermodynamic relations and sum rules are fulfilled.

The author proposed recently a simplification of the full parquet approximation by summing only two singlet two-particle channels yielding most divergent diagrams in the critical region of the metal-insulator transition [14]. A generating functional for such an approximation was derived via the linked-cluster theorem. The question is whether also the unabridged parquet equations can be integrated to a generating thermodynamic potential. It is the aim of this paper to show how the Luttinger-Ward functional and the grand potential can be constructed for the complete parquet equations, i. e. with all two-particle irreducible diagrams, in lattice models of interacting electrons with a local (Hubbard) interaction.

There are three topologically inequivalent definitions of a two-particle irreducibility. It may be defined according to cutting electron-hole or electron-electron pair propagation, or according to cutting polarization bubbles shielding the electron-electron interaction. Each possibility defines a two-particle channel of multiple pair scatterings characterized by a different binding of independent variables in the vertex functions. A general two-particle quantity will be denoted in this paper as in Fig. 1. Each two-particle function carries three independent four-momenta and two spin indices. We generally denote the fermionic four-momenta  $k = (\mathbf{k}, i\omega_n)$  and the transferred bosonic ones as  $q = (\mathbf{q}, i\nu_m)$ , where  $\omega_n = (2n+1)\pi T$  and  $\nu_m = 2m\pi T$  are the respective Matsubara frequencies at temperature  $T = \beta^{-1}$ .

It is convenient to introduce a matrix notation in the spin indices to distinguish different two-particle channels in the parquet diagrams. We define a  $2 \times 2$  matrix for the generic two-particle function  $X_{\sigma\sigma'}$ :

$$\hat{X} = \begin{pmatrix} X_{\uparrow\uparrow} & X_{\uparrow\downarrow} \\ X_{\downarrow\uparrow} & X_{\downarrow\downarrow} \end{pmatrix}. \quad (1)$$

We speak about singlet and triplet contributions to a two-particle function if the spins of the involved fermions are antiparallel or parallel, respectively.

We define three matrix multiplication schemes for two-particle quantities

$$[\hat{X} \bullet \hat{Y}]_{\sigma\sigma'}(k, k'; q) = \frac{1}{\beta\mathcal{N}} \sum_{q''} X_{\sigma\sigma'}(k, k'; q'') Y_{\sigma\sigma'}(k + q'', k' + q''; q - q''), \quad (2a)$$

$$[\hat{X} \circ \hat{Y}]_{\sigma\sigma'}(k, k'; q) = \frac{1}{\beta\mathcal{N}} \sum_{q''} X_{\sigma\sigma'}(k, k' + q''; q - q'') Y_{\sigma\sigma'}(k + q - q'', k'; q''), \quad (2b)$$

$$[\hat{X} \star \hat{Y}]_{\sigma\sigma'}(k, k'; q) = \frac{1}{\beta\mathcal{N}} \sum_{\sigma'' k''} X_{\sigma\sigma''}(k, k''; q) Y_{\sigma''\sigma'}(k'', k'; q) \quad (2c)$$

representing summations in the three inequivalent two-particle channels, electron-hole, ( $eh$ ), electron-electron, ( $ee$ ), and interaction, ( $U$ ), respectively. We see that the variables of the two-particle functions are convoluted differently in inequivalent channels. Note that only the interaction channel mixes the singlet and triplet contributions.

We decompose the full two-particle Green function into a sum of always reducible and irreducible projections onto each inequivalent channel

$$\mathcal{K}_{\sigma\sigma'}(k, k'; q) = \mathcal{K}_{\sigma\sigma'}^{\alpha}(k, k'; q) + I_{\sigma\sigma'}^{\alpha}(k, k'; q) \quad (3)$$

where  $\alpha = eh, ee, U$  refers to a two-particle channel.

The parquet diagrams can at best be represented graphically. Having in mind the above introduced notation and a general convention that double primed variables are summed over, we can write

$$\text{Diagram (4a)} \quad (4a)$$

$$\text{Diagram (4b)} \quad (4b)$$

$$\text{Diagram (4c)} \quad (4c)$$

We labeled only the intermediate four-momenta on the right-hand side of the parquet equations since the endpoints are the same as those on the left-hand side.

Equations (4) are the Bethe-Salpeter equations and constitute one set of the full parquet algebra. To complete it we must add relations connecting the two-particle reducible,  $\mathcal{K}^\alpha$ , and irreducible with two-particle bubbles,  $\Lambda^\alpha$ , functions. To this purpose we introduced horizontal ( $eh$  and  $ee$ ) and vertical ( $U$ ) two-particle bubbles

$$G_{\sigma\sigma'}^{(2)h}(k, k'; q) = G_\sigma(k + q)G_{\sigma'}(k' + q) , \quad G_{\sigma\sigma'}^{(2)v}(k, k'; q) = G_{\sigma'}(k')G_{\sigma'}(k' + q) . \quad (5)$$

The functions  $\Lambda^\alpha$  can then be defined as [15]

$$\Lambda_{\sigma\sigma'}^\alpha(k, k'; q) = \left[ U\delta_{\sigma', -\sigma} + \Delta I_{\sigma\sigma'}(k, k'; q) + \sum_{\alpha'' \neq \alpha} \mathcal{K}_{\sigma\sigma'}^{\alpha''}(k, k'; q) \right] G_{\sigma\sigma'}^{(2)\alpha}(k, k'; q) \quad (6)$$

where  $\Delta I_{\sigma\sigma'}$  is a sum of all Feynman diagrams simultaneously irreducible in each two-particle channel. The sum contains only higher-order diagrams where three and more particles are multiply interconnected. In the usual treatment with only two-particle multiple scatterings this irreducible part is neglected. We hence put  $\Delta I_{\sigma\sigma'} = 0$ .

The parquet algebra for the two-particle function is now complete. Equations (4) and (6) are to be integrated to a generating functional. We first construct two functionals being integrals of equations (4) and (6). We can exclude the reducible parts  $\mathcal{K}^\alpha$  from (4) and the irreducible ones  $\Lambda^\alpha$  from (6). If we denote the generating functionals for (4) and (6)  $\Phi_\Lambda$  and  $\Phi_K$  respectively, we demand

$$\mathcal{K}^\alpha = \frac{\delta \Phi_\Lambda}{\delta \Lambda^{T\alpha}}, \quad \Lambda^\alpha = \frac{\delta \Phi_K}{\delta \mathcal{K}^{T\alpha}}, \quad (7)$$

where  $X_{\sigma\sigma'}^{Tv} = X_{\sigma'\sigma}^v$  and  $X_{\sigma\sigma'}^{Th} = X_{\sigma\sigma'}^h$  is a matrix transposition in the vertical and horizontal channels, respectively. The functional  $\Phi_\Lambda$  will be a sum of functionals generating each of the parquet equations in (4), i. e. each channel will have its own generating functional. Recalling the matrix notation and the multiplication rules (2) we can integrate each parquet equation to the respective generating functional

$$\Phi_\Lambda^{eh} = \frac{1}{2\beta^2 \mathcal{N}^2} \sum_{\sigma k} \sum_{\sigma' k'} \left[ \hat{\Lambda}^{eh} - \frac{1}{2} \hat{\Lambda}^{eh} \bullet \hat{\Lambda}^{eh} - \ln \left( 1 + \hat{\Lambda}^{eh} \bullet \right) \right]_{\sigma\sigma'} (k, k'; 0), \quad (8a)$$

$$\Phi_\Lambda^{ee} = \frac{1}{2\beta^2 \mathcal{N}^2} \sum_{\sigma k} \sum_{\sigma' k'} \left[ \hat{\Lambda}^{ee} - \frac{1}{2} \hat{\Lambda}^{ee} \circ \hat{\Lambda}^{ee} - \ln \left( 1 + \hat{\Lambda}^{ee} \circ \right) \right]_{\sigma\sigma'} (k, k'; 0), \quad (8b)$$

$$\Phi_\Lambda^U = \frac{1}{2\beta^2 \mathcal{N}^2} \sum_{\sigma} \sum_{kq} \left[ \hat{\Lambda}^U - \frac{1}{2} \hat{\Lambda}^U \star \hat{\Lambda}^U - \ln \left( 1 + \hat{\Lambda}^U \star \right) \right]_{\sigma\sigma} (k, k; q). \quad (8c)$$

The symbols  $\hat{\Lambda} \bullet$ ,  $\hat{\Lambda} \circ$ ,  $\hat{\Lambda} \star$  in the logarithms set the appropriate multiplication for the power-series definition of the matrix function.

It is much easier to integrate equation (6). It is straightforward to verify that

$$\begin{aligned} \Phi_K &= \frac{1}{2\beta^2 \mathcal{N}^2} \sum_{\sigma k} \sum_{\sigma' k'} \left[ U \hat{\delta} \hat{G}^{(2)h} \left( \bullet \hat{\mathcal{K}}^{eh} + \circ \hat{\mathcal{K}}^{ee} \right) \hat{G}^{(2)h} + \hat{\mathcal{K}}^{eh} \hat{G}^{(2)h} \bullet \hat{\mathcal{K}}^{ee} \hat{G}^{(2)h} \right]_{\sigma\sigma'} (k, k'; 0) \\ &+ \frac{1}{2\beta^2 \mathcal{N}^2} \sum_{\sigma} \sum_{kq} \left[ \left( U \hat{\delta} + \hat{\mathcal{K}}^{eh} + \hat{\mathcal{K}}^{ee} \right) \hat{G}^{(2)v} \star \hat{\mathcal{K}}^U \hat{G}^{(2)v} \right]_{\sigma\sigma} (k, k, q), \end{aligned} \quad (9)$$

where  $\hat{\delta}_{\sigma\sigma'} = \delta_{\sigma', -\sigma}$ . Note that when we multiply functions from the electron-hole and electron-electron channels the multiplication rule can be taken from either channel without influencing the result. Connecting the incoming and outgoing fermion lines in the product is uniquely determined by the charge conservation.

The Luttinger-Ward functional must finally be stationary with respect to first variations of the two-particle functions. This can be achieved by subtracting the mixed  $\text{tr} \hat{\Lambda} \hat{\mathcal{K}}$  term. Moreover, the functional must generate the self-energy via a variation with respect to the full one-electron propagator. We hence must add a purely one-electron part to the generating functional. The one-electron contribution is just a second-order term in the free-energy expansion. The parquet diagrams sum multiple scatterings of pairs of quasiparticles starting with the next term beyond second order. We choose the correct overall sign of the generating functional from the self-energy, e. g. in second order. The Luttinger-Ward functional for the parquet diagrams finally reads

$$\begin{aligned} \Phi[G; \Lambda, \mathcal{K}; U] &= \frac{1}{2\beta^2 \mathcal{N}^2} \sum_{\sigma} \sum_{kq} \left[ \hat{\Lambda}^U \star \hat{\mathcal{K}}^U \hat{G}^{(2)v} \right]_{\sigma\sigma} (k, k, q) \\ &+ \frac{1}{2\beta^2 \mathcal{N}^2} \sum_{\sigma k} \sum_{\sigma' k'} \left[ \left( -\frac{1}{2} U \hat{\delta} \hat{G}^{(2)h} \bullet U \hat{\delta} + \hat{\Lambda}^{eh} \bullet \hat{\mathcal{K}}^{eh} + \hat{\Lambda}^{ee} \circ \hat{\mathcal{K}}^{ee} \right) \hat{G}^{(2)h} \right]_{\sigma\sigma'} (k, k'; 0) \\ &- \Phi_K[G; \mathcal{K}; U] - \Phi_\Lambda^{eh}[\Lambda^{eh}] - \Phi_\Lambda^{ee}[\Lambda^{ee}] - \Phi_\Lambda^U[\Lambda^U] \end{aligned} \quad (10)$$

where we introduced in each functional its explicit dependence on the variational one- and two-particle functions and the bare interaction  $U$ .

The grand potential generating the complete thermodynamics of the parquet approximation is constructed from the Luttinger-Ward functional and a free-electron term with the Hartree contribution [13]. We hence add also the particle densities as additional variational parameters and obtain for a fixed  $\mu_\sigma = \mu + \sigma h$  where  $\mu$  is a chemical potential and  $h$  an external magnetic field

$$\begin{aligned} \frac{1}{\mathcal{N}} \Omega[n_\uparrow, n_\downarrow; \Sigma, G; \Lambda, \mathcal{K}] = & -\frac{1}{\beta \mathcal{N}} \sum_{\sigma n, \mathbf{k}} e^{i\omega_n 0^+} \left\{ \ln [i\omega_n + \mu_\sigma - \epsilon(\mathbf{k}) - U n_{-\sigma} - \Sigma_\sigma(\mathbf{k}, i\omega_n)] \right. \\ & \left. + G_\sigma(\mathbf{k}, i\omega_n) \Sigma_\sigma(\mathbf{k}, i\omega_n) \right\} - U n_\uparrow n_\downarrow + \Phi[G; \Lambda, \mathcal{K}; U] . \end{aligned} \quad (11)$$

Here  $n_\uparrow, n_\downarrow; \Sigma, G; \Lambda, \mathcal{K}$  are independent variables and complex functions. Their physical values are chosen from stationarity of the grand potential (11) with respect to variations of its independent variables/functions. There are three pairs of Legendre conjugate variational variables in (11). The Hartree parameters  $n_\uparrow$  and  $n_\downarrow$ , the one-electron functions  $\Sigma_\sigma(k)$  and  $G_\sigma(k)$ , and finally the two-particle vertex functions  $\Lambda_{\sigma\sigma'}^\alpha(k, k'; q)$  and  $\mathcal{K}_{\sigma\sigma'}^\alpha(k, k'; q)$ .

The stationarity equations for the Hartree parameters lead to a one-electron sum rule

$$n_\sigma = \frac{1}{\beta \mathcal{N}} \sum_{n, \mathbf{k}} \frac{e^{i\omega_n 0^+}}{i\omega_n + \mu_\sigma - \epsilon(\mathbf{k}) - U n_{-\sigma} - \Sigma_\sigma(\mathbf{k}, i\omega_n)} . \quad (12)$$

The stationarity equations for the one-particle functions, i. e. the one-electron propagator and the self-energy, measuring in this formulation corrections to the Hartree approximation, read

$$G_\sigma(\mathbf{k}, i\omega_n) = [i\omega_n + \mu_\sigma - \epsilon(\mathbf{k}) - U n_{-\sigma} - \Sigma_\sigma(\mathbf{k}, i\omega_n)]^{-1} , \quad (13a)$$

$$\Sigma_\sigma(k) = -\frac{U}{\beta^2 \mathcal{N}^2} \sum_{\sigma'} \sum_{k' q} G_\sigma(k+q) \mathcal{K}_{\sigma\sigma'}(k+q, k'; -q) G_{\sigma'}(k') G_{\sigma'}(k'+q) . \quad (13b)$$

Finally, the stationarity equations for the two-particle functions are just the parquet equations (4) and (6) with  $\Delta I_{\sigma\sigma'} = 0$ . The full two particle Green function is a sum of a second-order contribution and the reducible parts from each two-particle inequivalent channel

$$\mathcal{K}_{\sigma\sigma'}(k, k'; q) = U \delta_{\sigma', -\sigma} + \mathcal{K}_{\sigma\sigma'}^{eh}(k, k'; q) + \mathcal{K}_{\sigma\sigma'}^{ee}(k, k'; q) + \mathcal{K}_{\sigma\sigma'}^U(k, k'; q) . \quad (14)$$

Thermodynamics of the parquet approximation is completely determined by (8)-(14) into which appropriate auxiliary external sources are introduced. All thermodynamic relations are hence fulfilled in the parquet summation scheme. Moreover due to the full self-consistency in the one- and two-particle propagators, the parquet approximation obeys momentum and energy conservation laws [13]. However, it is unclear whether the parquet approximation conserves also charge as the only generator of the electrostatic energy. The electrostatic energy (due to the Coulomb interaction  $U$ ) of the solutions must entirely be generated by the actual distribution of charge carriers (electron density  $n$ ). This conservation law induces a Ward-like identity connecting a variation in the electrostatic energy with a variation in the mass distribution [14]. The identity relates a thermodynamic correlation function defined as a derivative of the one-electron density with respect to an auxiliary field (electro-chemical potential) and the full two-particle Green function  $\mathcal{K}_{\uparrow\downarrow}$  from (14).

The correlation function describing the variation of the electrostatic energy can be derived from the generating functional if we introduce a small perturbation of the local Coulomb interaction  $U \rightarrow U + \delta U(q)$ . We define the correlation function as

$$\begin{aligned} \mathcal{C}(q) &= \left. \frac{\delta \Phi[G; \Lambda, \mathcal{K}; U]}{\delta U(q)} \right|_{\delta U=0} \\ &= -\frac{1}{2\beta^2 \mathcal{N}^2} \sum_{kk'} G_{\uparrow\downarrow}^{(2)}(k, k'; q) \{ \mathcal{K}_{\uparrow\downarrow}(k, k'; -q) + \mathcal{K}_{\uparrow\downarrow}(k, k'; q) \} G_{\uparrow\downarrow}^{(2)}(k, k'; -q) . \end{aligned} \quad (15)$$

The static part of this dynamic correlation function is important for the thermodynamic stability of a particular solution (phase). This function must be negative in a stable phase

$$\mathcal{C}(\mathbf{q}, 0) < 0 \quad (16)$$

as a consequence of a repulsive character of the Coulomb interaction [14]. The correlation function  $\mathcal{C}(\mathbf{q}, 0)$  is from definition connected with the magnetic susceptibility  $\chi(\mathbf{q})$  and the compressibility  $\kappa(\mathbf{q})$

$$\mathcal{C}(\mathbf{q}, 0) = \frac{1}{\mathcal{N}} \sum_{\mathbf{ij}} e^{-i(\mathbf{R}_i - \mathbf{R}_j)\mathbf{q}} [\langle n_{i\uparrow} n_{j\downarrow} \rangle - \langle n_{i\uparrow} \rangle \langle n_{j\downarrow} \rangle] \equiv \frac{T}{4} [\kappa(\mathbf{q}) - \chi(\mathbf{q})] , \quad (17)$$

which together with (16) indicates that in tight-binding models with a local electrostatic interaction the spin fluctuations are stronger than the charge ones.

Equality (17) is rigorously fulfilled in an exact solution. The Ward identity due to conservation of sources of the electrostatic energy demands that sums of both sides of (17) over momenta must equal, otherwise there are spurious sources of the electrostatic potential in the system. Approximate theories with only one-electron renormalizations break equality (17) qualitatively, i. e. the left- and right-hand sides lead to incompatible phase diagrams and it is unclear how to define two-particle functions consistently. The only way to improve upon this discrepancy is to introduce dynamical vertex (two-particle) renormalizations. It is clear that the exact equality in (17) can be achieved in self-consistent theories only in a complete solution where arbitrarily large clusters of particles are involved. In order to produce unambiguous results for two-particle correlation functions it must be required from approximate theories that the left- and right-hand sides of (17) generate the same divergences and lead to equivalent phase diagrams.

Although the parquet diagrams contain an advanced dynamical renormalization of the vertex functions, skeleton three-particle diagrams are not involved. Higher-order skeleton diagrams are generated by derivatives of the self-energy (13b) with respect to the electro-chemical potential when constructing the susceptibility or the compressibility. However, skeleton diagrams beyond the parquet approximation are fully renormalized in the two-particle sector, i. e. they use appropriate projections of the full two-particle Green function  $\mathcal{K}_{\sigma\sigma'}$  instead of the bare interaction  $U$ . The lowest-order contribution to  $\Delta I_{\sigma\sigma'}$  is shown in Fig. 2. Unlike other simpler approximations the two-particle functions in the parquet approach can show only *integrable singularities*. A singularity can arise in one or more two-particle channels simultaneously. Only one bosonic variable, the conserved transfer four-momentum in the multiple scattering events, is relevant in each channel. It is for the electron-hole, electron-electron, and interaction channel  $k - k'$ ,  $k + k' + q$ , and  $q$ , respectively. The divergences in two-particle functions are smeared out (regularized) whenever one integrates over the variables in which the singularity arises. We realize that three-particle skeleton diagrams as in Fig. 2 contain integrals over all relevant two-particle four-momenta simultaneously and are hence finite. Unless *qualitatively new* singularities arise in three-particle (or higher-order) skeleton diagrams, the parquet approximation is stable with respect to fluctuations and the left- and right-hand sides of (17) lead to the same instabilities and hence to qualitatively the same phase diagram. Appearance of new singularities in three-particle and higher-order skeleton diagrams, not seen at the two-particle level, would indicate in the renormalization-group language that higher-order effective interactions are relevant. Neither the Bethe ansatz nor the renormalization group suggest relevance of interactions beyond the two-particle ones.

Parquet diagrams are a systematic approximation summing all two-particle irreducible Feynman graphs with fully renormalized one-electron propagators. The idea behind this summation scheme is to replace an expansion in the bare interaction strength  $U$  by skeleton diagrams with renormalized two-particle functions. This is a realization of the field-theoretic renormalization of perturbation theory close to critical points. Parquet approximation is the simplest theory with two explicit vertex functions containing minimally one-loop renormalization of the critical behavior. However, one cannot reduce the singular two-particle Green function to a static running coupling constant as in classical phase transitions. As well as we cannot reduce renormalized perturbation theory to a weak-coupling loop expansion. Due to a complicated structure of dynamical fluctuations at quantum phase transitions in itinerant systems, one has to keep dynamical two-particle Green functions in the parquet equations and their extensions. Parquet diagrams offer a comprehensive basis for a reliable assessment of the critical behavior at quantum phase transitions with singularities in two-particle Green functions.

Extensions of the parquet approximation contain only skeleton diagrams contributing to  $\Delta I_{\sigma\sigma'}$  free of one- and two-particle insertions. The interaction strength is everywhere replaced by the two-particle Green function. The parquet approximation offers an overall reliable solution whenever the *norm* of the full two-particle function  $\|\mathcal{K}G^{(2)}\| \ll 1$ . Since the parquet equations allow only for integrable

singularities in the two-particle Green functions, their solution may obey the reliability criterion even at intermediate and strong coupling. At least  $\|\mathcal{KG}^{(2)}\| < 1$  is guaranteed in the parquet approximation. This feature makes it an appropriate tool for studying the transition region between the weak-coupling, Fermi-liquid solution and the strong-coupling, local-moment phase in the lattice models with an effective local (Hubbard) interaction.

To conclude, we integrated the parquet diagrams for two-particle Green functions to a generating Luttinger-Ward functional. We demonstrated that the parquet summation scheme belongs to thermodynamically consistent and conserving  $\Phi$ -derivable approximations in Baym's sense if the two particle functions are generated from appropriate functional derivatives of the self-energy. The difference between the thermodynamic (variational derivatives) and diagrammatic definitions of two-particle functions (Aslamazov-Larkin diagrams [16]), reflecting deviations from exact identity (17), are expected to vanish in leading asymptotic order at two-particle critical points.

Although the explicit grand potential does not help to resolve the complicated algebra of dynamical variables coupled in the parquet equations, it enables one to control systematic expansions beyond and eventual simplifications within the parquet approximation. The parquet diagrams and their derivatives have been presented in a new light from which the role, relevance, and systematics of the parquet-type graphs have become more transparent.

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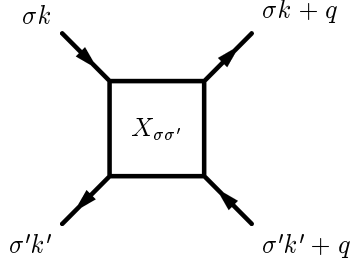


FIG. 1. Generic two-particle function with three independent four-momenta and a defined order of incoming and outgoing fermions.

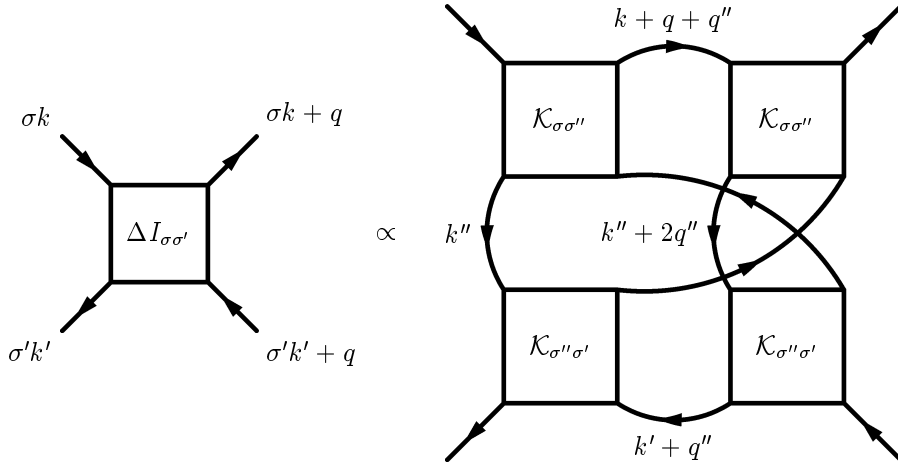


FIG. 2. Lowest-order contribution not included in the parquet approximation. The bare interaction is replaced by the full two-particle function from the parquet approximation. Double primed variables are summation indices.